Math 2114 : Lecture 7 : Matrix Inverses

Definition: Let A be a $n \times n$ matrix. If there exists a $n \times n$ matrix B such that

$$AB = I_n \quad \text{and} \quad BA = I_n \tag{1}$$

In general, $AB \neq BA$

we call the matrix A invertible and the matrix B an inverse of A.

Example 1: Consider the matrices

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \tag{2}$$

Show that $AB = \overset{I_2}{I_n}$ and $BA = \overset{I_2}{I_n}$. Conclude that A is invertible with inverse B.

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$BA = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

By definition, A is invertible with inverse B.

$AB = I_n$ ✓ and $BA = I_n$ ✓

Note 1: By symmetry, If the matrix A is invertible with inverse B, then the matrix B is also invertible with inverse A.

$BA = I_n$ ✓ and $AB = I_n$ ✓

> *Theorem 1 (Poole 3.13):* Let A be a $n \times n$ matrix. If B is a $n \times n$ matrix such that either $AB = I_n$ or $BA = I_n$ then $AB = BA = I_n$ and hence A is invertible with inverse B.

Example 2: Consider the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \tag{3}$$

Show that A is invertible with inverse B.

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

By theorem 1, $BA = I_3$ and thus A is inventable with inverse B.

> *Theorem 2 (Poole 3.6):* Let A be a $n \times n$ matrix. If A is invertible, then its inverse is unique.

Note 2: If A is invertible, we denote its unique inverse A^{-1}. $\cancel{\frac{1}{A}}$ $\leftarrow$ Do not write like this

Note 3: If A is invertible, then A^{-1} is invertible with inverse $(A^{-1})^{-1} = A$ by symmetry. **This note is theorem 3.9 part (a) in Poole**.

Proof: Suppose ① $AB = I_n$ and $BA = I_n$

② $AC = I_n$ and $CA = I_n$

$$B = I_n B = (CA)B = C(AB) = C(I_n) = C$$

$\uparrow$ assosiativity

Thus $B = C$.